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Construction of Supergravity Backgrounds with a Dilaton Field

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Abstract

A new class of non-compact Kähler backgrounds accompanied by a non-constant dilaton field is constructed as a supergravity solution. It is interpreted as a complex line bundle over a base manifold comprising of a combination of arbitrary coset spaces, and also includes the case of Calabi-Yau manifolds. The resulting backgrounds have $U(1)$ isometry. We consider $N=2$ supersymmetric σ-models on them, and derive a non-Kählerian solution by $U(1)$ duality transformation, which preserves $N=2$ supersymmetry.

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1 Introduction

Non-critical string theories with a non-trivial dilaton field have been extensively studied in order to analyse dynamics of two-dimensional gravity and Liouville theory. The Liouville theory is equivalent to the linear dilaton system, where the configuration of the dilaton has rich structures in dynamics of string theories.

For the purpose of generalizing this dilaton system, we construct supergravity backgrounds interpreted as a complex line bundle over a base manifold comprising of a combination of arbitrary coset spaces. They have a non-constant dilaton field in general. Our study includes the discussion in Ref.[1], which solves the Einstein equation to get Ricci flat Kähler metrics of a complex line bundle over coset spaces.

The backgrounds found here necessarily have $U(1)$ isometry. By considering an $N = 2$ supersymmetric σ-model on them, we can perform $U(1)$ duality transformation.[2] The duality symmetry originating from such an isometry can be understood as replacing a chiral superfield with a twisted chiral superfield by a Legendre transformation. The duality symmetry also relates backgrounds which have geometrically different properties but correspond to the same conformal field theory. Therefore, by this transformation, we can get another σ-model, which is equivalent to the original one as a conformal field theory. The resulting dual background consists of a metric, a dilaton, and an anti-symmetric tensor field. As a result it is no longer Kählerian possessing non-trivial torsion.

This paper is organized as follows: In Section 2, we explain a complex line bundle over coset spaces, and then derive a non-compact Kähler background with a non-trivial dilaton as a solution of supergravity. In Section 3, we give several explicit examples of such backgrounds. In Section 4, we consider the $N = 2$ supersymmetric σ-model on the background obtained in Section 2. By performing a duality transformation on it, we construct a non-Kählerian background. Section 5 is devoted to summaries and conclusions.

2 Complex Line Bundle over Coset Spaces

In this section we construct a new class of Kähler metrics with a non-trivial dilaton. We consider a canonical line bundle L over an arbitrary Kähler coset space M, and couple the metric G_{MN} to the dilaton Φ. The base manifold M is a direct product of N Kähler coset spaces G_a/H_a $(a = 1, \cdots, N)$

$$M = (G_1/H_1) \times (G_2/H_2) \times \cdots \times (G_N/H_N)\,, \tag{2.1}$$

and we assume the Kähler potential K of L is a function of the G-invariant X defined by[1]

$$X = \log\left(|\sigma|^2 e^{\sum_{a=1}^N h_a\Psi_a}\right) = \log|\sigma|^2 + \hat{\Psi}, \quad \hat{\Psi} = \sum_{a=1}^{N} h_a\Psi_a, \tag{2.2}$$

where Ψ_a is a Kähler potential corresponding to G_a/H_a, h_a is a real positive constant, and σ is a complex coordinate of a fiber of L.

The technology for finding the Kähler potential for an arbitrary coset space G_a/H_a is developed in Refs.[3, 4], using the supersymmetric nonlinear realization [5, 6, 7]. Let $G_a^{\mathbf{C}}$

be a complexification of G_a, and $\hat{H}_a$ be some complex group including $H_a^{\mathbf{C}}$ (a complexification of H_a) such that Lie algebra $\hat{\mathcal{H}}_a$ of $\hat{H}_a$ is complex isotropy algebra. There is a homeomorphism $G_a/H_a \cong G_a^{\mathbf{C}}/\hat{H}_a$. Complex coordinates φ^i parametrize the coset space $G_a^{\mathbf{C}}/\hat{H}_a$, and the representative of the coset is given by

$$\xi_a(\varphi) = \exp(i\varphi \cdot \mathcal{X}), \quad \{\mathcal{X}\} : \text{generators of } \mathcal{G}_a^{\mathbf{C}} - \hat{\mathcal{H}}_a. \tag{2.3}$$

The Kähler potential of G_a/H_a is represented as

$$\Psi_a(\varphi, \bar{\varphi}) = \sum_{\alpha_a=1}^{k_a} v_{\alpha_a} \log \det_{\eta_{\alpha_a}} \xi_a^{\dagger}(\bar{\varphi})\, \xi_a(\varphi), \tag{2.4}$$

where k_a is the dimension of torus in H_a, v_{α_a} is a real positive constant, and $\det_{\eta_{\alpha_a}}$ is the determinant of the subspace projected by a projection matrix η_{α_a}. This yields the Kähler metric and the Ricci tensor

$$g_{i\bar{j}}^{(a)} = \partial_i \partial_{\bar{j}} \Psi_a, \quad R_{i\bar{j}}^{(a)} = -\partial_i \partial_{\bar{j}} \log \det g_{k\bar{\ell}}^{(a)}, \tag{2.5}$$

where $\partial_i = \frac{\partial}{\partial \varphi_i}$. With a suitable choice of v_α, any Kähler coset space is shown to become a Kähler-Einstein manifold

$$R_{i\bar{j}}^{(a)} = h_a g_{i\bar{j}}^{(a)}. \tag{2.6}$$

Note that h_a in Eq.(2.2) is the same as this coefficient. Eqs.(2.5) and (2.6) give the expression of the determinant with a holomorphic function "*hol.*"

$$\det g_{i\bar{j}}^{(a)} = e^{-h_a \Psi_a} \cdot |hol.|^2. \tag{2.7}$$

The total space L is parametrized by the coordinates σ, φ_a^i $(a = 1, 2, \cdots, N)$, where the index i runs through the dimension of each coset space as $i = 1, \cdots, \dim_C G_a/H_a$, but we often omit "$a$" if there is no confusion. The metric G_{MN} of L is given in terms of the Kähler potential K as $G_{\mu\nu} = G_{\bar{\mu}\bar{\nu}} = 0$ and

$$G_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K = \begin{pmatrix} G_{\sigma\bar{\sigma}} & G_{\sigma\bar{j}} \\ G_{i\bar{\sigma}} & G_{i\bar{j}} \end{pmatrix}, \tag{2.8}$$

whose components are represented as

$$G_{\sigma\bar{\sigma}} = K'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \bar{\sigma}}, \quad G_{i\bar{j}} = K'' \frac{\partial X}{\partial \varphi^i} \frac{\partial X}{\partial \bar{\varphi}^{\bar{j}}} + K' \frac{\partial^2 K}{\partial \varphi^i \partial \bar{\varphi}^{\bar{j}}}, \tag{2.9}$$

$$G_{i\bar{\sigma}} = K'' \frac{\partial X}{\partial \bar{\sigma}} \frac{\partial X}{\partial \varphi^i}, \quad G_{\sigma\bar{j}} = K'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \bar{\varphi}^{\bar{j}}}, \tag{2.10}$$

where $K' = dK/dX$, and the Ricci-tensor is

$$R_{\mu\nu} = R_{\bar{\mu}\bar{\nu}} = 0, \quad R_{\mu\bar{\nu}} = -\partial_\mu \partial_{\bar{\nu}} \log \det G_{\kappa\bar{\lambda}}. \tag{2.11}$$

In order that this background provides a consistent string theory, the two-dimensional σ-model has to be conformal invariant. At one-loop level, conformal invariance requires the following equations of motion for the background fields G_{MN} and Φ:

$$R_{\mu\bar{\nu}} = -2\partial_\mu \partial_{\bar{\nu}} \Phi, \tag{2.12}$$

$$R_{\mu\nu} = -2\nabla_\mu \nabla_\nu \Phi. \tag{2.13}$$

Let us solve these differential equations to obtain the metric and the dilaton under the ansatz $K = K(X)$, $\Phi = \Phi(X)$. Eqs. (2.11) and (2.12) imply

$$\det G_{\mu\bar{\nu}} = e^{2\Phi}|hol.|^2. \tag{2.14}$$

From the expression

$$\begin{aligned}\det G_{\mu\bar{\nu}} &= \det G_{\sigma\bar{\sigma}} \cdot \det(G_{i\bar{j}} - G_{i\bar{\sigma}}\frac{1}{G_{\sigma\bar{\sigma}}}G_{\sigma\bar{j}}) \\ &= e^{-X}K'' \cdot (K')^D \cdot |hol.|^2 = \frac{1}{D+1}e^{-X}\frac{d}{dX}(K')^{D+1} \cdot |hol.|^2,\end{aligned} \tag{2.15}$$

Eq.(2.14) becomes

$$2\Phi = -X + \log\left[\frac{1}{D+1}\frac{d}{dX}(K')^{D+1}\right], \tag{2.16}$$

where we defined

$$D = \sum_{a=1}^{N} \dim_C G_a/H_a. \tag{2.17}$$

Also Eqs. (2.11) and (2.13) imply

$$\nabla_\mu \nabla^{\bar{\nu}}\Phi = \partial_\mu(G^{\bar{\nu}\lambda}\partial_\lambda\Phi) = 0. \tag{2.18}$$

From the expressions

$$G^{\bar{\nu}\mu} = \begin{pmatrix} G^{\bar{\sigma}\sigma} & G^{\bar{\sigma}i} \\ G^{\bar{j}\sigma} & G^{\bar{j}i} \end{pmatrix} = \begin{pmatrix} \frac{|\sigma|^2}{K''} + \frac{|\sigma|^2}{K'}\sum_a h_a g^{\bar{j}i}_{(a)}\partial_{\bar{j}}\Psi_a \cdot \partial_i\Psi_a & -\frac{\bar{\sigma}}{K'}\sum_a \partial_{\bar{j}}\Psi_a \cdot g^{\bar{j}i}_{(a)} \\ -\frac{\sigma}{K'}\sum_a g^{\bar{j}i}_{(a)} \cdot \partial_i\Psi_a & \frac{1}{K'}\sum_a h_a^{-1} g^{\bar{j}i}_{(a)} \end{pmatrix}, \tag{2.19}$$

Eq.(2.18) becomes

$$\partial_\mu(G^{\bar{\sigma}\lambda}\partial_\lambda\Phi) = \bar{\sigma}\partial_\mu[(K'')^{-1}\partial_X\Phi] = 0, \tag{2.20}$$

so we get the equation

$$\partial_X[(K'')^{-1}\partial_X\Phi] = 0. \tag{2.21}$$

With the definition of the variable Y

$$Y = K', \tag{2.22}$$

this differential equation can be solved to yield the dilaton

$$2\Phi = -aY + b, \quad a, b = const., \quad a \geq 0. \tag{2.23}$$

In the case of $a = 0$, the background with a constant dilaton becomes a Calabi-Yau manifold due to Ricci flatness and our discussion coincides with Ref.[1]. In the case of $a \neq 0$, the background corresponds to the linear dilaton system with respect to Y.

For $a \neq 0$ we substitute (2.23) into (2.16) and acquire

$$e^{X+b} = \int_0^Y dY\, Y^D e^{aY} + C, \tag{2.24}$$

with an integration constant C. By using the formula

$$\int_0^y dx\, x^n e^x = (-1)^n n! \left[-1 + e^y \sum_{m=0}^{n} \frac{(-y)^m}{m!} \right], \tag{2.25}$$

Eq.(2.24) can be rewritten into

$$Be^X = -A + e^{aY} \sum_{m=0}^{D} \frac{(-aY)^m}{m!}, \tag{2.26}$$

$$A = 1 - C \cdot (-1)^D a^{D+1} \cdot \frac{1}{D!}, \quad B = e^b \cdot (-1)^D a^{D+1} \cdot \frac{1}{D!}. \tag{2.27}$$

Thus we obtain the metric of the total space L. Using Eqs.(2.22) and (2.24), and setting $\sigma = e^{\frac{r}{2}+i\theta}$ the metric takes the form

$$\begin{aligned} ds^2 &= G_{MN} dx^M dx^N = 2G_{\mu\bar{\nu}} dx^\mu dx^{\bar{\nu}} \\ &= \frac{a}{2f(aY)} dY^2 + \frac{2f(aY)}{a} \left[\mathrm{Im}(d\varphi^i \partial_i \hat{\Psi}) + d\theta \right]^2 + 2Y \sum_{a=1}^{N} h_a ds_a^2, \end{aligned} \tag{2.28}$$

with

$$ds_a^2 = g^{(a)}_{i\bar{j}} d\varphi^i d\bar{\varphi}^{\bar{j}}, \tag{2.29}$$

$$f(aY) = aY' = (-1)^D D! (aY)^{-D} \left[-Ae^{-aY} + \sum_{m=0}^{D} \frac{(-aY)^m}{m!} \right], \tag{2.30}$$

$$d\theta = \mathrm{Im}(d \log \sigma). \tag{2.31}$$

The scalar curvature of this background metric can be calculated as

$$R = 2a(1 - aY') = 2a(1 - f(aY)). \tag{2.32}$$

We can evaluate asymptotic behavior of the function $f(aY)$

$$f(aY) \approx 1 - \frac{D}{aY} - A\frac{(-1)^D D!}{(aY)^D} e^{-aY} + \mathcal{O}\left(\frac{1}{(aY)^2}\right) \quad (aY \gg 1),$$

$$f(aY) \approx (-1)^D D! \frac{1-A}{(aY)^D} e^{-aY} + \frac{aY}{D+1} + \mathcal{O}\left((aY)^2\right) \quad (aY \approx 0).$$

Therefore in the limit of $Y \to \infty$, R approaches 0. At $Y = 0$, the space is regular for $C = 0$ ($A = 1$), but it has a singularity for $C \neq 0$ ($A \neq 1$).

For $a = 0$, Eq.(2.24) leads to

$$\frac{f(aY)}{a} = Y' = \frac{1}{D+1} Y + C \cdot Y^{-D}, \tag{2.33}$$

and by plugging this into Eq.(2.28) we can get the expression of the metric.

Moreover the central charge deficit δc provided by this background is determined by the β-function of the dilaton field as

$$\delta c = c - \frac{3}{2} \cdot 2(D+1) = 3\left[2(\nabla\Phi)^2 - \nabla^2\Phi\right]. \tag{2.34}$$

By inserting the dilaton (2.23) this becomes

$$\delta c = \left[af(aY) + af'(aY) + \frac{D}{Y}f(aY)\right] = 3a, \tag{2.35}$$

and we obtain the central charge

$$c = 3(D+1+a). \tag{2.36}$$

3 Examples

As discussed in the previous section, once we choose the base coset space M, we can construct the metrics of the complex line bundle over the base manifold; calculating $\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi})$ and $\partial_i\partial_{\bar{j}}\hat{\Psi}$ yields background metrics (2.28). In this section, we give some concrete examples by using Kähler potentials for various coset spaces.

In the following subsections, we consider only the case that the base manifold consists of a single coset space, since the discussion on the case containing more than one Kähler coset spaces is straightforward.

3.1 Hermitian Symmetric Spaces

To begin with, let us construct metrics for the complex line bundles over hermitian symmetric spaces. For these case, several examples of the Kähler potentials for the base coset spaces are explicitly derived in Refs.[3, 8], and the G-invariant X is expressed by [1]

$$X = \log|\sigma|^2 + \hat{\Psi}, \quad \hat{\Psi} = h\Psi, \quad \Psi = v\log\det_{\eta}\xi^\dagger\xi \equiv v\log\Xi, \quad h = \frac{1}{2v}\,2\,\tilde{h}(G), \tag{3.1}$$

where $\tilde{h}(G)$ is the dual Coxeter number of G, and we normalized the generators of the fundamental representation as

$$\mathrm{Tr}(T^AT^B) = \delta^{AB}. \tag{3.2}$$

We summarize their results in Table.1. From these results we calculate the metrics for the complex line bundles over the hermitian symmetric spaces.

3.1.1 Projective Space: $CP^{N-1} = SU(N)/[SU(N-1)\times U(1)]$

By using Ξ and $\tilde{h}(G)$ in Table.1, we can calculate

$$\partial_i\partial_{\bar{j}}\hat{\Psi} = h\partial_i\partial_{\bar{j}}\Psi = hg_{i\bar{j}} = N\left[\frac{1}{\Xi}\delta_{ij} - \frac{1}{\Xi^2}\varphi^j\bar{\varphi}^{\bar{i}}\right], \tag{3.3}$$

$$\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi}) = N\cdot\frac{i}{2}\frac{1}{\Xi}\left(d\bar{\varphi}^{\bar{i}}\varphi^i - d\varphi^i\bar{\varphi}^{\bar{i}}\right) \equiv -NA, \tag{3.4}$$

Table 1: Summary of hermitian symmetric spaces G/H.

G/H	$\tilde{h}(G)$	Ξ	$\dim_C G/H$
CP^{N-1}	N	$1+\lvert\varphi^i\rvert^2$	$N-1$
		$(i=1,2,\cdots,N-1)$	
Q^{N-2}	$N-2$	$1+\lvert\varphi^i\rvert^2+\frac{1}{4}(\varphi^i)^2(\bar{\varphi}^{\bar{j}})^2$	$N-2$
		$(i=1,2,\cdots,N-2)$	
$G_{N,M}$	N	$\det(\mathbf{1}_M+\varphi^\dagger\varphi)$	$M(N-M)$
		$(\varphi_{Aa}: A=1,2,\cdots,N-M,\ a=1,2,\cdots,M)$	
$Sp(N)/U(N)$	$N+1$	$\det(\mathbf{1}_N+\varphi^\dagger\varphi)$	$\frac{1}{2}N(N+1)$
		$(\varphi_{ab}: a,b=1,\cdots,N,\ \varphi_{ab}=\varphi_{ba},\ a\leq b)$	
$SO(2N)/U(N)$	$2N-2$	$\det(\mathbf{1}_N+\varphi^\dagger\varphi)$	$\frac{1}{2}N(N-1)$
		$(\varphi_{ab}: a,b=1,\cdots,N,\ \varphi_{ab}=-\varphi_{ba},\ a<b)$	
$E_6/[SO(10)\times U(1)]$	12	$1+\lvert\varphi_\alpha\rvert^2+\frac{1}{8}\lvert\varphi_\alpha(C\sigma_A^\dagger)^{\alpha\beta}\varphi_\beta\rvert^2$	16
		$(\alpha=1,2,\cdots,16,\ A=1,2,\cdots,10)$	
$E_7/[E_6\times U(1)]$	18	$1+\lvert\varphi^i\rvert^2+\frac{1}{4}\lvert\Gamma_{ijk}\varphi^j\varphi^k\rvert^2+\frac{1}{36}\lvert\Gamma_{ijk}\varphi^i\varphi^j\varphi^k\rvert^2$	27
		$(i=1,2,\cdots,27)$	

and we get the metric

$$ds^2=\frac{1}{2Y'}dY^2+2Y'\left[d\theta-NA\right]^2+2NY\,ds_{FS}^2, \tag{3.5}$$

where A is a connection 1-form on CP^{N-1} and ds_{FS}^2 is the Fubini-Study metric. After rescaling $Y\to Y/N$, we obtain

$$ds^2=\frac{1}{2NY'}dY^2+\frac{2Y'}{N}\left[d\theta-NA\right]^2+2Y\,ds_{FS}^2, \quad \Phi=-\frac{a}{2N}Y+const. \tag{3.6}$$

This corresponds to a solution of $\beta^G_{MN}=0$ under $U(N)$ isometry found in Refs.[2, 9, 10] as a generalization of the two-dimensional black hole background [11, 12].

For the other cases in Table 1, we obtain the metrics by calculating $\partial_i\partial_{\bar{j}}\hat{\Psi}$ and $\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi})$. The results are collected in the following:

3.1.2 Quadratic Space: $Q^{N-2}=SO(N)/[SO(N-2)\times U(1)]$

$$\partial_i\partial_{\bar{j}}\hat{\Psi}=(N-2)\left[\frac{1}{\Xi}(\delta_{ij}+\varphi^i\bar{\varphi}^{\bar{j}})-\frac{1}{\Xi^2}(\bar{\varphi}^{\bar{i}}+\frac{1}{2}\varphi^i(\bar{\varphi}^{\bar{k}})^2)(\varphi^j+\frac{1}{2}(\varphi^l)^2\bar{\varphi}^{\bar{j}})\right], \tag{3.7}$$

$$\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi})=(N-2)\cdot\frac{i}{2}\frac{1}{\Xi}\left[d\bar{\varphi}^{\bar{i}}(\varphi^i+\frac{1}{2}(\varphi^k)^2\bar{\varphi}^{\bar{i}})-d\varphi^i(\bar{\varphi}^{\bar{i}}+\frac{1}{2}\varphi^i(\bar{\varphi}^{\bar{k}})^2)\right], \tag{3.8}$$

3.1.3 Grassmanian: $G_{N,M} = SU(N)/[SU(N-M) \times U(M)]$

$$\frac{\partial^2 \hat{\Psi}}{\partial \varphi_{Aa} \partial \varphi^*_{Bb}} = N(\mathbf{1}_M + \varphi^\dagger \varphi)^{-1}_{ab} \left[\mathbf{1}_{(N-M)} - \varphi(\mathbf{1}_M + \varphi^\dagger \varphi)^{-1} \varphi^\dagger \right]_{BA}, \tag{3.9}$$

$$\mathrm{Im}\left(d\varphi_{Aa} \frac{\partial}{\partial \varphi_{Aa}} \hat{\Psi} \right) = N \cdot \frac{i}{2} (\mathbf{1}_M + \varphi\varphi^\dagger)^{-1}_{ab} (d\varphi^\dagger \varphi - \varphi^\dagger d\varphi)_{ba}. \tag{3.10}$$

3.1.4 $Sp(N)/U(N)$ and $SO(2N)/U(N)$

Setting $\epsilon = -1$ for $Sp(N)/U(N)$, and $\epsilon = +1$ for $SO(2N)/U(N)$,

$$\frac{\partial^2 \hat{\Psi}}{\partial \varphi_{ab} \partial \varphi^*_{cd}} = \tilde{h}(G) \left(1 - \frac{1}{2}\delta_{ab}\right) \left(1 - \frac{1}{2}\delta_{cd}\right) \Big[(\mathbf{1}_N + \varphi^\dagger \varphi)^{-1}_{bd} \{ \mathbf{1}_N - \varphi(\mathbf{1}_N + \varphi^\dagger \varphi)^{-1} \varphi^\dagger \}_{ca}$$

$$- \epsilon (\mathbf{1}_N + \varphi^\dagger \varphi)^{-1}_{bc} \{ \mathbf{1}_N - \varphi(\mathbf{1}_N + \varphi^\dagger \varphi)^{-1} \varphi^\dagger \}_{da} + (a \leftrightarrow b,\ c \leftrightarrow d) \Big], \tag{3.11}$$

$$\mathrm{Im}\left(d\varphi_{ab} \frac{\partial \hat{\Psi}}{\partial \varphi_{ab}} \right) = \tilde{h}(G) \cdot \frac{i}{2} \left(1 - \frac{1}{2}\delta_{ab}\right) \Big[d\varphi^\dagger_{ba} \{ (\varphi(\mathbf{1}_N + \varphi^\dagger \varphi)^{-1})_{ab} - \epsilon (\varphi(\mathbf{1}_N + \varphi^\dagger \varphi)^{-1})_{ba} \}$$

$$- d\varphi_{ab} \{ ((\mathbf{1}_N + \varphi^\dagger \varphi)^{-1} \varphi^\dagger)_{ba} - \epsilon ((\mathbf{1}_N + \varphi^\dagger \varphi)^{-1} \varphi^\dagger)_{ab} \} \Big]. \tag{3.12}$$

3.1.5 Exceptional Group: $E_6/[SO(10) \times U(1)]$

With φ_α's chiral superfields belonging to an $SO(10)$ Weyl spinor representation, σ_A's $SO(10)$ γ-matrices in the Weyl spinor basis, and C a charge conjugation matrix [13],

$$\partial_\alpha \partial_{\bar{\beta}} \hat{\Psi} = 12 \Big[\frac{1}{\Xi} \{ \delta_{\alpha\beta} + \frac{1}{2} (\sigma^A C^\dagger \bar{\varphi})^\beta (C \sigma_A{}^\dagger \varphi)^\alpha \}$$

$$- \frac{1}{\Xi^2} \{ \bar{\varphi}_{\bar{\alpha}} + \frac{1}{4} (C \sigma^\dagger_A \varphi)^\alpha (\bar{\varphi} \sigma^A C^\dagger \bar{\varphi}) \} \{ \varphi_\beta + \frac{1}{4} (\sigma^A C^\dagger \bar{\varphi})^\beta (\varphi C \sigma^\dagger_A \varphi) \} \Big], \tag{3.13}$$

$$\mathrm{Im}(d\varphi^\alpha \partial_\alpha \hat{\Psi}) = 12 \cdot \frac{i}{2} \frac{1}{\Xi} \Big[d\bar{\varphi}_{\bar{\beta}} \{ \varphi_\beta + \frac{1}{4} (\varphi C \sigma_A{}^\dagger \varphi)(\sigma^A C^\dagger \bar{\varphi})^\beta \}$$

$$- d\varphi_\alpha \{ \bar{\varphi}_{\bar{\alpha}} + \frac{1}{4} (\bar{\varphi} \sigma^A C^\dagger \bar{\varphi})(C \sigma^\dagger_A \varphi)^\alpha \} \Big]. \tag{3.14}$$

3.1.6 Exceptional Group: $E_7/[E_6 \times U(1)]$

With φ_i's chiral superfields belonging to the fundamental representation **27** of E_6, Γ_{ijk} the rank 3 anti-symmetric tensor, and Γ^{ijk} its complex conjugate [13],

$$\partial_i\partial_{\bar{j}}\hat{\Psi} = 18\Bigg[\frac{1}{\Xi}\{\delta_{ij} + (\Gamma_{ikl}\varphi^l)(\Gamma^{jkm}\bar{\varphi}_{\bar{m}}) + \frac{1}{4}(\Gamma_{ikl}\varphi^k\varphi^l)(\Gamma^{jmn}\bar{\varphi}_{\bar{m}}\bar{\varphi}_{\bar{n}})\}$$
$$-\frac{1}{\Xi^2}\{\bar{\varphi}_{\bar{i}} + \frac{1}{2}(\Gamma_{ikl}\varphi^l)(\Gamma^{kmn}\bar{\varphi}_{\bar{m}}\bar{\varphi}_{\bar{n}}) + \frac{1}{12}(\Gamma_{ikl}\varphi^k\varphi^l)(\Gamma^{mnp}\bar{\varphi}_{\bar{m}}\bar{\varphi}_{\bar{n}}\bar{\varphi}_{\bar{p}})\}$$
$$\times\{\varphi^j + \frac{1}{2}(\Gamma_{klm}\varphi^l\varphi^m)(\Gamma^{jkn}\bar{\varphi}_{\bar{n}}) + \frac{1}{12}(\Gamma_{klm}\varphi^k\varphi^l\varphi^m)(\Gamma^{jnp}\bar{\varphi}_{\bar{n}}\bar{\varphi}_{\bar{p}})\}\Bigg], \tag{3.15}$$

$$\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi}) = 18\cdot\frac{i}{2}\frac{1}{\Xi}\Bigg[d\bar{\varphi}_{\bar{i}}\{\varphi^i + \frac{1}{2}(\Gamma^{ilm}\bar{\varphi}_{\bar{l}})(\Gamma_{mnp}\varphi^n\varphi^p) + \frac{1}{12}(\Gamma^{ilm}\bar{\varphi}_{\bar{l}}\bar{\varphi}_{\bar{m}})(\Gamma_{jnp}\varphi^j\varphi^n\varphi^p)\}$$
$$-d\varphi^i\{\bar{\varphi}_{\bar{i}} + \frac{1}{2}(\Gamma_{ilm}\varphi^l)(\Gamma^{mnp}\bar{\varphi}_{\bar{n}}\bar{\varphi}_{\bar{p}}) + \frac{1}{12}(\Gamma_{ilm}\varphi^l\varphi^m)(\Gamma^{jnp}\bar{\varphi}_{\bar{j}}\bar{\varphi}_{\bar{n}}\bar{\varphi}_{\bar{p}})\}\Bigg]. \tag{3.16}$$

3.2 Non-Symmetric Spaces

Next we consider $SU(l+m+n)/S[U(l)\times U(m)\times U(n)]$ derived in Ref.[3], as a example of non-symmetric spaces. It is known that there exists two kinds of complex structures on this coset space[14, 15]. For simplicity, we explicitly construct the metric for the case of $l=m=n=1$. In this case the two structures lead to the same model and the Kähler potential is expressed with the complex coordinates $\varphi^A, \varphi^B, \varphi^C$ as[1]

$$\hat{\Psi} = h\Psi, \quad \Psi = v_1\log\Xi_1 + v_2\log\Xi_2, \quad h = \frac{2}{v_1} = \frac{2}{v_2}, \tag{3.17}$$
$$\Xi_1 = 1 + |\varphi^C|^2 + |\varphi^B + \frac{1}{2}\varphi^A\varphi^C|^2, \quad \Xi_2 = 1 + |\varphi^A|^2 + |\varphi^B - \frac{1}{2}\varphi^A\varphi^C|^2. \tag{3.18}$$

We can compute the metric in the same way as above

$$\partial_i\partial_{\bar{j}}\hat{\Psi} = 2\Bigg[\frac{1}{{\Xi_1}^2}F(\varphi^A,\varphi^B,\varphi^C,\bar{\varphi}^{\bar{A}},\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{C}}) + \frac{1}{{\Xi_2}^2}F(\varphi^C,-\varphi^B,\varphi^A,\bar{\varphi}^{\bar{C}},-\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{A}})\Bigg], \tag{3.19}$$
$$\mathrm{Im}(d\varphi^i\partial_i\hat{\Psi}) = 2\cdot\frac{i}{2}\Bigg[\frac{1}{\Xi_1}G(\varphi^A,\varphi^B,\varphi^C,\bar{\varphi}^{\bar{A}},\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{C}}) + \frac{1}{\Xi_2}G(\varphi^C,-\varphi^B,\varphi^A,\bar{\varphi}^{\bar{C}},-\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{A}})\Bigg], \tag{3.20}$$

where $i = A, B, C$ and

$$F(\varphi^A,\varphi^B,\varphi^C,\bar{\varphi}^{\bar{A}},\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{C}})$$
$$= \begin{pmatrix} d\bar{\varphi}^{\bar{A}} \\ d\bar{\varphi}^{\bar{B}} \\ d\bar{\varphi}^{\bar{C}} \end{pmatrix}^T \begin{pmatrix} \frac{1}{4}|\varphi^C|^2(1+|\varphi^C|^2) & \frac{1}{2}\bar{\varphi}^{\bar{C}}(1+|\varphi^C|^2) & \frac{1}{4}\varphi^A\bar{\varphi}^{\bar{C}} - \frac{1}{2}\varphi^B(\bar{\varphi}^{\bar{C}})^2 \\ \frac{1}{2}\varphi^C(1+|\varphi^C|^2) & 1+|\varphi^C|^2 & \frac{1}{2}\varphi^A - \varphi^B\bar{\varphi}^{\bar{C}} \\ \frac{1}{4}\bar{\varphi}^{\bar{A}}\varphi^C - \frac{1}{2}\bar{\varphi}^{\bar{B}}(\varphi^C)^2 & \frac{1}{2}\bar{\varphi}^{\bar{A}} - \bar{\varphi}^{\bar{B}}\varphi^C & 1+\frac{1}{4}|\varphi^A|^2 + |\varphi^B|^2 \end{pmatrix} \begin{pmatrix} d\varphi^A \\ d\varphi^B \\ d\varphi^C \end{pmatrix}, \tag{3.21}$$

$$G(\varphi^A,\varphi^B,\varphi^C,\bar{\varphi}^{\bar{A}},\bar{\varphi}^{\bar{B}},\bar{\varphi}^{\bar{C}})$$
$$= d\bar{\varphi}^{\bar{C}}\varphi^C + d(\bar{\varphi}^{\bar{B}} + \frac{1}{2}\bar{\varphi}^{\bar{A}}\bar{\varphi}^{\bar{C}})(\varphi^B + \frac{1}{2}\varphi^A\varphi^C) - d\varphi^C\bar{\varphi}^{\bar{C}} - d(\varphi^B + \frac{1}{2}\varphi^A\varphi^C)(\bar{\varphi}^{\bar{B}} + \frac{1}{2}\bar{\varphi}^{\bar{A}}\bar{\varphi}^{\bar{C}}). \tag{3.22}$$

If l, m, n take other values, we can also obtain an explicit formula for the metric by the similar procedure.

4 $U(1)$ Duality Transformation

The Kählerian backgrounds we have found are solutions of the equations of motion. Since they have a $U(1)$ isometry, i.e. their metric G_{MN} does not depend on θ explicitly, we can construct their dual spaces by replacing one chiral superfield with a twisted chiral superfield[2]. The dual space in general contains non-trivial torsion and the resulting metric is no longer Kählerian. In this section, we construct the dual of the solution with a non-trivial dilaton obtained in Section 2.

In our case the $N = 2$ superspace action is determined by the following Kähler potential

$$K = K(Z + \bar{Z}, \Phi_i, \bar{\Phi}_{\bar{j}}), \tag{4.1}$$

where Z and Φ_i are chiral superfields whose lowest components are $z = \frac{r}{2} + i\theta = \log \sigma$ and φ_i respectively. In order to get the dual potential $\tilde{K}$, we perform the Legendre transformation as follows:

$$\tilde{K}(\Psi + \bar{\Psi}, \Phi_i, \bar{\Phi}_{\bar{j}}) = K(Z + \bar{Z}, \Phi_i, \bar{\Phi}_{\bar{j}}) - (Z + \bar{Z})(\Psi + \bar{\Psi}), \tag{4.2}$$

$$\frac{\partial K}{\partial R} = \Psi + \bar{\Psi}, \quad R = Z + \bar{Z}, \tag{4.3}$$

with a twisted chiral superfield Ψ containing its lowest component ψ. Now the independent variables are substituted with $\psi, \bar{\psi}, \varphi_i, \bar{\varphi}_{\bar{j}}$. The metric and anti-symmetric tensor are obtained by writing down the bosonic part of the superspace action [16],

$$\tilde{G}_{\mu\bar{\nu}} = \begin{pmatrix} & -\tilde{K}_{\psi\bar{\psi}} & & \\ -\tilde{K}_{\psi\bar{\psi}} & & & \\ & & & \tilde{K}_{i\bar{j}} \\ & & \tilde{K}_{i\bar{j}} & \end{pmatrix}, \quad B_{\mu\bar{\nu}} = \begin{pmatrix} & & & \tilde{K}_{\psi\bar{j}} \\ & & \tilde{K}_{i\bar{\psi}} & \\ & -\tilde{K}_{i\bar{\psi}} & & \\ -\tilde{K}_{\psi\bar{j}} & & & \end{pmatrix}, \tag{4.4}$$

whose components are given by

$$\tilde{K}_{\psi\bar{\psi}} = \frac{\partial^2 \tilde{K}}{\partial\psi\partial\bar{\psi}} = -\frac{\partial X}{\partial\psi} = -\frac{a}{f(aY)}, \quad \tilde{K}_{i\bar{j}} = \frac{\partial^2 \tilde{K}}{\partial\varphi^i\partial\bar{\varphi}^{\bar{j}}} = (\psi + \bar{\psi}) \cdot \partial_i\partial_{\bar{j}}\hat{\Psi}, \tag{4.5}$$

$$\tilde{K}_{\psi\bar{j}} = \frac{\partial^2 \tilde{K}}{\partial\psi\partial\bar{\varphi}^{\bar{j}}} = \partial_{\bar{j}}\hat{\Psi}, \quad \tilde{K}_{i\bar{\psi}} = \frac{\partial^2 \tilde{K}}{\partial\bar{\psi}\partial\varphi^i} = \partial_i\hat{\Psi}. \tag{4.6}$$

In the above calculation, we used the relation derived from (2.22) and (4.3)

$$Y = \psi + \bar{\psi}, \tag{4.7}$$

and the expression of the derivative obtained from (2.26)

$$\frac{\partial X}{\partial\psi} = \frac{a}{f(aY)}. \tag{4.8}$$

Therefore we get the dual metric

$$d\tilde{s}^2 = \frac{2a}{f(aY)} d\psi d\bar{\psi} + 2Y\, \partial_i \partial_{\bar{j}} \hat{\Psi}\, d\varphi^i d\bar{\varphi}^{\bar{j}}. \tag{4.9}$$

The dual dilaton is discussed in Ref.[17] to yield

$$\tilde{\Phi} = \Phi - \frac{1}{2} \log\left(2K_{rr}\right), \tag{4.10}$$

where r is the lowest component of $R = Z + \bar{Z}$, and this turns out to be

$$\tilde{\Phi} = -\frac{1}{2} \log\left[e^{aY} \frac{f(aY)}{a}\right] + const. \tag{4.11}$$

The dual dilaton $\tilde{\Phi}$ depends on Y nonlinearly and gives nontrivial backgrounds. We can calculate the field strength $H_{LMN} = \partial_L B_{MN} + \partial_M B_{NL} + \partial_N B_{LM}$ and its non-zero components are represented by $\hat{\Psi}$

$$H_{\psi i \bar{j}} = -\partial_i \partial_{\bar{j}} \hat{\Psi}, \quad H_{\bar{\psi} i \bar{j}} = \partial_i \partial_{\bar{j}} \hat{\Psi}. \tag{4.12}$$

These are collected into the 3-form H

$$H = \frac{1}{6} H_{LMN}\, dx^L \wedge dx^M \wedge dx^N = -2\, d(\mathrm{Im}\psi) \wedge \sum_{a=1}^{N} h_a J_a, \tag{4.13}$$

where J_a is the Kähler form of the coset G_a/H_a:

$$J_a = i g^{(a)}_{i\bar{j}} d\varphi^i \wedge d\bar{\varphi}^{\bar{j}}. \tag{4.14}$$

One can easily check $\tilde{G}_{MN}$, $\tilde{\Phi}$, and H_{LMN} derived here satisfy the background field equations $\beta^{\tilde{G}}_{MN} = 0$ and $\beta^{B}_{MN} = 0$. The central charge in this dual model is evaluated from $\beta^{\tilde{\Phi}}$ as

$$\begin{aligned} \delta\tilde{c} = \tilde{c} - \frac{3}{2} \cdot 2(D+1) &= 3\left[2(\nabla\tilde{\Phi})^2 - \nabla^2\tilde{\Phi} - \frac{1}{12} H_{LMN} H^{LMN}\right] \\ &= 3\left[a f(aY) + a f'(aY) + \frac{D}{Y} f(aY)\right] = 3a. \end{aligned} \tag{4.15}$$

This result is equal to (2.35) completely, and consistent with the fact that the duality transformation preserves the conformal invariance of the theory.

The scalar curvature has a complicated form as

$$\begin{aligned} \tilde{R} &= 2\left[a\left(f''(aY) - \frac{f'(aY)^2}{f(aY)}\right) + \frac{3D}{2a}\frac{f(aY)}{Y^2} - \frac{D^2}{a}\frac{f(aY)}{Y^2} + \frac{D}{Y}\right] \\ &= 2\left[a\left(1 - \frac{1}{f(aY)}\right) + \frac{2D}{Y} + \frac{5D}{2a}\frac{f(aY)}{Y^2} - \frac{D^2}{a}\frac{f(aY)}{Y^2}\right]. \end{aligned} \tag{4.16}$$

Let us discuss the dependence on Y of the dilaton $\tilde{\Phi}$ and the curvature $\tilde{R}$. The Y is limited to the region where the string coupling constant $g = \exp\tilde{\Phi} > 0$. For $C > 0$ ($A > 1$

Table 2: Dependence on Y of $\tilde{\Phi}$ and $\tilde{R}$. The "$*$" means diverging to $+\infty$ or $-\infty$ i.e. curvature singularity.

$C > 0$

Y	0	$\cdots$	∞
$f(aY)$	$+\infty$	$\cdots$	1
$\tilde{\Phi}$	$-\infty$		$-\infty$
$\tilde{R}$	$*$		0

$C = 0$

Y	0	$\cdots$	∞
$f(aY)$	0	$\cdots$	1
$\tilde{\Phi}$	$+\infty$		$-\infty$
$\tilde{R}$	$*$		0

$C < 0$

Y	Y_0	$\cdots$	∞
$f(aY)$	0	$\cdots$	1
$\tilde{\Phi}$	$+\infty$		$-\infty$
$\tilde{R}$	$*$		0

for odd D, $A < 1$ for even D) and $C = 0$ ($A = 1$), $f(aY) > 0$ in $Y > 0$, and so Y's range is $0 < Y < \infty$. For $C < 0$ ($A < 1$ for odd D, $A > 1$ for even D), there exists a unique $Y_0 > 0$ satisfying $f(aY_0) = 0$, and $f(aY)$ monotonically increases in $Y > Y_0$. Hence we consider $Y_0 < Y < \infty$. In Table.2, we explain how $f(aY)$, $\tilde{\Phi}$, and $\tilde{R}$ vary depending on Y. The space has one singular point and becomes flat as $Y \to \infty$ in each case.

Here we comment on the case of $a = 0$. Substituting (2.33) into (4.9) and (4.11), we obtain the expression of the dual metric and dilaton. The dual dilaton is no longer a constant. The curvature is also calculated from (4.16) and its dependence on Y can be discussed as above.

5 Conclusions

We derived non-compact Kähler backgrounds as a solution of the Einstein equation. The backgrounds are interpreted as a complex line bundle over a base manifold comprising of a combination of arbitrary coset spaces. They have a non-constant dilaton field in general, and come to be Calabi-Yau manifolds when a parameter vanishes. If the base coset space is designated, we can give an explicit formula of the total space metric.

Furthermore, we obtained a non-Kählerian solution by performing the duality transformation with respect to the $U(1)$ isometry. The dual background is equivalent to the original one as a conformal field theory, but has a different form as an $N = 2$ supersymmetric σ-model.

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